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THE HILBERT SERIES OF PRIME PI RINGS

BY

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ABSTRACT

We study the Hilbert series of finitely generated prime PI algebras. We show that given such an algebra A there exists some finite dimensional subspace V of A which contains 1_A and generates A as an algebra such that the Hilbert series of A with respect to the vector space V is a rational function.

1. Introduction

Given a field k, a k-algebra A, and a finite dimensional k-subspace of A which contains 1_A and generates A as a k-algebra, we define the **Hilbert series** of A with respect to V to be

$$H_A(t) := 1 + \sum_{n=1}^{\infty} \dim(V^n/V^{n-1})t^n. \tag{1.1}$$

The Hilbert series is sometimes called the **Poincaré series**. One is especially interested in the case that $H_A(t)$ is a rational function of t. The following theorem shows the significance of this.

THEOREM 1.1: *Let A be a finitely generated k-algebra with rational Hilbert series with respect to some generating subspace V.*

- *If $H_A(t)$ has radius of convergence 1, then $\mathrm{GKdim}(A)$ is equal to the order of the pole of $H_A(t)$ at $t = 1$. In particular, the GK dimension is a nonnegative integer; moreover, $\dim(V^n)$ is a polynomial in n for all n sufficiently large.*
- *If the radius of convergence of $H_A(t) < 1$, then A has exponential growth.*

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- *If $H_A(t)$ has radius of convergence greater than 1, then $H_A(t)$ is a polynomial and A is finite dimensional over k.*

Proof: See Theorem 12.6.2 on page 175 of [6]. ∎

There are several examples of algebras which have rational Hilbert series. On page 176 of [6], the following list of algebras which have rational Hilbert series is given.

- A finitely generated commutative algebra.
- Enveloping algebras of finitely generated Lie algebras.
- Finitely presented monomial algebras ([5], [12]).
- Trace rings of generic matrices (page 204 of [4]).
- Generic PI algebras ([1]).
- The group algebra of a finitely generated abelian-by-finite group ([2]).
- The group algebra of the first Heisenberg group ([8]).
- A Noetherian, connected graded algebra which is fully bounded ([10]).
- A Noetherian, connected graded algebra which has finite global dimension ([11]).

The proofs of the first two items follow from material in Chapter 7 of [6]. In this paper, we prove the following result.

THEOREM 1.2: *Let A be a finitely generated prime PI k-algebra. Then A has rational Hilbert series with respect to some generating subspace V.*

We note that this result is not necessarily true if A is not prime. A simple construction due to Warfield (see Theorem 2.9 of [6]) shows this. For example, we let I be the ideal of $k\{x, y\}$ generated by $(x)^3$ and xy^ix for i not a perfect square and consider the algebra

$$A = k\{x, y\}/I.$$

Notice that A has a basis given by

$$\{y^i x y^{j^2} x y^\ell \mid i, j, \ell \geq 0\} \cup \{y^i x y^j \mid i, j \geq 0\} \cup \{y^i \mid i \geq 0\}.$$

Thus if V is the vector space spanned by 1 and the images of x and y in A, then

$$\begin{aligned}\dim(V^n) = & \sum_{0\le j\le\sqrt{n}} \binom{n-j^2}{2} + \binom{n+1}{2} + (n+1) \\ \ge & \sum_{\sqrt{n/2}\le j\le\sqrt{n}} \binom{n-j^2}{2} \\ \ge & \sum_{\sqrt{n/2}\le j\le\sqrt{n}} \binom{n/2}{2} \\ \ge & (\sqrt{n}-\sqrt{n/2}-1)n(n-2)/8 \\ \ge & n^{5/2}/30 \quad \text{for all } n \text{ sufficiently large.}\end{aligned}$$

A straightforward estimate shows that

$$\dim(V^n) \le (\sqrt{n}+1)n(n-1)/2 + \binom{n+1}{2} + (n+1) \le 2n^{5/2}$$

for all $n \ge 2$. Thus A has GK dimension 2.5. Since A is a homomorphic image of $k\{x,y\}/(x)^3$, which satisfies the identity $(x_1x_2 - x_2x_1)^3$, we see that A must be PI. From Theorem 1.1, we see that A cannot have a rational Hilbert series with respect to any generating vector space V.

Stafford [9] has constructed a finitely generated PI algebra A along with generating vector spaces V and W such that A has a rational Hilbert series with respect to V but not with respect to W. We follow notes of Lorenz in giving his construction. Let

$$S = \begin{pmatrix} k+(z) & k[x,y,z] \\ (z) & k[x,y,z] \end{pmatrix} \subseteq M_n(k[x,y,z]).$$

Clearly S is PI since it is a subring of a matrix ring over a commutative ring. Also, it is easy to check that S is finitely generated as a k-algebra. Define

$$\epsilon_i := \begin{cases} 1 & \text{if } i \text{ is a perfect square; and} \\ 0 & \text{otherwise.} \end{cases}$$

We create the vector space

$$U := \sum_{i=0}^{\infty} kzx^i\epsilon_i + kzy^i(1-\epsilon_i).$$

We define

$$I := \begin{pmatrix} (z^2, zxy) + U & (z,xy) \\ (z^2, zxy) & (z,xy) \end{pmatrix} \subseteq S.$$

Notice that I is an ideal of S. Let $A = S/I$ and let $e_{i,j}$ denote the matrix with 1 in the $(i,j)^{\text{th}}$ entry and zeros everywhere else. Stafford defines V to be the image in A of the k-vector space

$$ke_{1,1} + ke_{1,2} + kze_{2,1} + ke_{2,2} + kxe_{2,2} + kye_{2,2}$$

and defines W to be $V + \overline{kzxe_{2,1}}$. He shows that for $n \geq 2$,

$$\dim(V^n) = \dim(V^{n-1}) + 7$$

and

$$\dim(W^n) = \dim(W^{n-1}) + 7 - \epsilon_{n-2}.$$

If $H_{A,V}(t)$ and $H_{A,W}(t)$ denote the Hilbert series of A with respect to V and with respect to W respectively, then

$$H_{A,V}(t) = 1 + 5t + 7t^2/(1-t)$$

while

$$H_{A,W}(t) = 1 + 6t + 7t^2/(1-t) - \sum_{i \geq 0} x^{i^2+2}.$$

Observe that

$$\sum_{i \geq 0} x^{i^2+2}$$

is not rational by Theorem 1.1 and hence $H_{A,W}(t)$ is not rational.

2. Background results on Gröbner bases

Given a field k and a finitely generated k-algebra A, we write

$$A = k\{t_1, \ldots, t_m\}/I,$$

where I is the kernel of the map from the free algebra $k\{t_1, \ldots, t_m\}$ onto A. Let V denote the image of the k-vector space $k + kt_1 + \cdots + kt_m$ in A. Computing a (noncommutative) **Gröbner basis** for the ideal I allows one to compute the dimension of V^n for all values of n. We put a **degree lexicographical ordering** on the variables by declaring that

$$t_1 < t_2 < \cdots < t_m.$$

The set of words in $t_1, \ldots, t_m$ forms a k-basis for the free algebra $k\{t_1, \ldots, t_m\}$. Given a nonzero element $a \in I$, we express a as a k-linear combination of words in $t_1, \dots, t_m$. We define $\mathrm{in}(a)$, the **initial monomial** of a, to be the lexicographically greatest word with a nonzero coefficient in the expression for a as a k-linear combination of words in $t_1, \dots, t_m$. We define $\mathrm{in}(0) = 0$ and we define

$$\mathrm{in}(I) := \langle\{\mathrm{in}(a) | \; a \in I\}\rangle. \tag{2.2}$$

We have the following theorem.

PROPOSITION 2.1: *Let*

$$V = k + kt_1 + \cdots + kt_m + I \subseteq k\{t_1, \dots, t_m\}/I$$

and let

$$W = k + kt_1 + \cdots + kt_m + \mathrm{in}(I) \subseteq k\{t_1, \dots, t_m\}/\mathrm{in}(I).$$

Then $\dim(V^n) = \dim(W^n)$ *for all* $n \geq 1$.

Proof: The proof is similar to that of Theorem 15.3 in [3]. ∎

COROLLARY 2.1: *Suppose* $k\{t_1, \dots, t_m\}/I$ *has the property that* $\mathrm{in}(I)$ *is finitely generated. Then* $k\{t_1, \dots, t_m\}/I$ *has rational Hilbert series.*

Proof: By the preceding proposition, $k\{t_1, \dots, t_m\}/I$ has the same Hilbert series as $k\{t_1, \dots, t_m\}/\mathrm{in}(I)$. By [5], [12] we have that this Hilbert series must be rational. ∎

3. Proofs

Let A be a finitely generated prime PI k-algebra over some field k. Then it is well-known that the GK dimension of A is some nonnegative integer d and for any generating subspace V of A, there exists a positive constant $C = C(V)$ such that

$$\dim V^n \leq Cn^d.$$

Furthermore, we have a polynomial ring

$$k[x_1, \dots, x_d] \subseteq Z(A). \tag{3.3}$$

Extend the set $\{x_1, \dots, x_d\}$ to a generating set for A, say

$$\{x_1, \dots, x_d, y_1, \dots, y_m\}. \tag{3.4}$$

Let V be the k-span of $\{1, x_1, \ldots, x_d, y_1, \ldots, y_m\}$. Then there is some positive constant C such that $\dim(V^n) \leq Cn^d$ for all $n \geq 1$. Let N be a positive integer satisfying

$$N > 2Cd!. \tag{3.5}$$

Let w be a word in $y_1, \ldots, y_m$. Let w_i denote the subword of w consisting of the first i letters of w (we take w_0 to be 1). Let

$$\mathfrak{T} = \left\{ w \,\middle|\, \sum_{i=0}^{\text{length}(w)} w_i P_i(x_1, \ldots, x_d) = 0, P_{\text{length}(w)} \neq 0 \right\}. \tag{3.6}$$

Lemma 3.1: *Let w be a word of length at least N. Then some initial subword of w is in $\mathfrak{T}$.*

Proof: Notice that

$$\mathfrak{S}_n := \{w_i v \mid v \text{ a monomial in } x_1, \ldots, x_d \text{ of degree at most } n-i\} \subseteq V^n \tag{3.7}$$

cannot be a linearly independent set for n sufficiently large. Indeed, if this were the case, then V^n would have dimension at least

$$\sum_{i=0}^{N} \binom{n-i+d}{d} \sim (N+1)n^d/d!, \quad \text{as } n \to \infty,$$

since there are $\binom{n-i+d}{d}$ monomials in $x_1, \ldots, x_d$ of degree $n-i$. Since

$$(N+1)n^d/d! > 2\dim(V^n)$$

for all n, we see that for n sufficiently large, the set $\mathfrak{S}_n$ is linearly dependent. Hence there exist polynomials $p_i(x_1, \ldots, x_d)$, not all zero, such that

$$\sum_{i=0}^{N} w_i p_i(x_1, \ldots, x_n) = 0.$$

Hence every word in $y_1, \ldots, y_m$ of length at least N has some initial subword in $\mathfrak{T}$. ∎

It follows that there exists a finite subset $\mathfrak{S}$ of $\mathfrak{T}$ such that every word in $y_1, \ldots, y_m$ of length at least N has some word in $\mathfrak{S}$ as an initial subword. Let $w \in \mathfrak{S}$, and as before let w_i denote the subword of w consisting of the first i letters of w, where i ranges from 0 to the length of w. Then we have

$$\sum_{i=0}^{\text{length}(w)} w_i P_{w,i}(x_1, \ldots, x_d) = 0, \tag{3.8}$$

with $P_{w,\mathrm{length}(w)} \neq 0$. Let

$$q(x_1,\ldots,x_d) = \prod_{w\in\mathfrak{S}} P_{w,\mathrm{length}(w)}. \tag{3.9}$$

Since $q(x_1,\ldots,x_d) \in Z(A)$, we have that the nonnegative integer powers of $q(x_1,\ldots,x_d)$ form an Ore set which we call Ω. Let

$$B = \Omega^{-1}A. \tag{3.10}$$

We shall now show that B has rational Hilbert series with respect to some vector space and use this fact, along with the description of our vector space, to show that A has a rational Hilbert series.

THEOREM 3.1: *Let B be as in equation (3.10). Then B is finitely presented and there exists a vector space V such that B has a rational Hilbert series with respect to V.*

Proof: We define new variables $y_1',\ldots,y_m'$, which satisfy

$$y_i' = y_i q^{-1}, \quad \text{for } 1 \leq i \leq m, \tag{3.11}$$

where $y_1,\ldots,y_m$ are as in equation (3.4). We define a set $\mathfrak{S}'$ as follows. Given a word w in $y_1,\ldots,y_m$ in $\mathfrak{S}$, we let u denote the corresponding word in $y_1',\ldots,y_m'$. We define $\mathfrak{S}'$ to be the set of all words in $y_1',\ldots,y_m'$ which correspond to some word in $\mathfrak{S}$. Let w be a word in $y_1,\ldots,y_m$ in $\mathfrak{S}$ and let u denote the corresponding word in $y_1',\ldots,y_m'$. Then $u = wq^{-\mathrm{length}(w)}$. As before, we let u_i denote the subword of u consisting of the first i letters of u. Multiplying both sides of equation (3.8) by $q^n/P_{w,\mathrm{length}(w)}$ and using the fact that $u_iq^i = w_i$, we see

$$\sum_{i=0}^{\mathrm{length}(u)} u_i q^{\mathrm{length}(w)-i} P_{w,i}/P_{w,\mathrm{length}(w)} = 0. \tag{3.12}$$

For each $w \in \mathfrak{S}$, we create new variables

$$\theta_{w,i} \quad \text{for } 1 \leq i < \mathrm{length}(w).$$

Consider the free algebra F on generators

(3.13)

$$\mathfrak{G} := \{T,S\} \cup \{\theta_{w,i} \mid w \in \mathfrak{S}, 1 \leq i < \mathrm{length}(w)\} \cup \{X_1,\ldots,X_d,Y_1,\ldots,Y_m\}.$$

We have a surjection Φ from F onto B given by

$$\begin{aligned}
&\Phi(T) = q(x_1,\ldots,x_d), \\
&\Phi(S) = 1/q(x_1,\ldots,x_d), \\
&\Phi(\theta_{w,i}) = q^{\mathrm{length}(w)-i}P_{w,i}/P_{w,\mathrm{length}(w)} && \text{for } w \in \mathfrak{S}, 1 \le i < \mathrm{length}(w), \\
&\Phi(X_i) = x_i' && \text{for } 1 \le i \le d, \\
&\Phi(Y_j) = y_j' && \text{for } 1 \le j \le m.
\end{aligned} \tag{3.14}$$

Let

$$I := \ker(\Phi). \tag{3.15}$$

For convenience, we define

$$\mathfrak{Z} := \{T,S\} \cup \{\theta_{w,i} \mid w \in \mathfrak{S}, 1 \le i < \mathrm{length}(w)\} \cup \{X_1,\ldots,X_d\}. \tag{3.16}$$

Note that $\Phi(\mathfrak{Z})$ is a subset of $Z(B)$. From equation (3.12), we have for $u \in \mathfrak{S}'$

$$u + \sum_{i=0}^{\mathrm{length}(w)-1} u_i\theta_{w,i} = 0 \mod I. \tag{3.17}$$

We shall now construct a Gröbner basis for our algebra B. We put a degree lexicographical ordering on our generating set by placing some ordering on the $\theta_{w,i}$ and declaring that

$$\theta_{w,i} < X_d \quad \text{for all } w \in \mathfrak{S}, 1 \le i < \mathrm{length}(w), \tag{3.18}$$

and declaring that

$$Y_1 > \cdots > Y_m > S > T > X_1 > \cdots > X_d. \tag{3.19}$$

We first observe that the elements of the set

$$\{az \mid z \in \mathfrak{Z}, a \in \mathfrak{G}, a > z\}$$

are all in in(I), since $za - az \in I$ for all $z \in \mathfrak{Z}$ and $a \in \mathfrak{G}$ with $a < z$. Hence any monomial can be written (mod in(I)) as vu, where u is a monomial in $Y_1,\ldots,Y_m$ and v is a monomial in elements of $\mathfrak{Z}$. Next observe that equation (3.17) gives that for any word u in $y_1',\ldots,y_m'$ in $\mathfrak{S}'$, the corresponding word in $Y_1,\ldots,Y_m$ is in in(I). Define $\mathfrak{S}''$ to be the set of words in $Y_1,\ldots,Y_m$ corresponding to the words in $y_1',\ldots,y_m'$ in $\mathfrak{S}'$. Define J to be the ideal generated by the elements of $\mathfrak{S}''$ and by the elements of the set

$$\{az \mid z \in \mathfrak{Z}, a \in \mathfrak{G}, a > z\};$$

that is,

$$J := \langle \mathfrak{S}'' \cup \{az \mid z \in \mathfrak{Z}, a \in \mathfrak{G}, a > z\}\rangle. \tag{3.20}$$

Since every word of length N has some initial subword in $\mathfrak{S}'$, we see that every monomial in $Y_1, \ldots, Y_m$ of length at least N is in $\mathrm{in}(I)$. Thus if a monomial w is in $\mathrm{in}(I) \setminus J$, then w must have degree at most N in $Y_1, \ldots, Y_d$. Hence the set $\mathfrak{M}$ of monomials in $Y_1, \ldots, Y_m$ which are not in $\mathrm{in}(I)$ is a finite set. Let $u \in \mathfrak{M}$. Consider the set of monomials, v, in elements of $\mathfrak{Z}$ such that $vu \in \mathrm{in}(I)$. The monomial ideal in $k[\mathfrak{Z}]$ generated by such v is of course finitely generated by, say, $v_{u,1}, \ldots, v_{u,m_u}$. We make the following claim.

Claim: $\mathrm{in}(I)$ *is generated by the finite set of monomials*

$$\mathfrak{S}'' \cup \{v_{u,i}u \mid u \in \mathfrak{M}, 1 \le i \le m_u\} \cup \{za \mid z \in \mathfrak{Z}, a \in \mathfrak{G}, z > a\}. \tag{3.21}$$

Proof: Let $r \in \mathrm{in}(I)$. We may assume that r is a monomial of the form vu, where u is a monomial in $Y_1, \ldots, Y_m$ and v is a monomial in elements of $\mathfrak{Z}$. If $u \notin \mathfrak{M}$, then some initial subword of u must be in $\mathfrak{S}''$ and we obtain the desired result. Hence $u \in \mathfrak{M}$. By construction, v is in the monomial ideal of $k[\mathfrak{Z}]$ generated by $v_{u,1}, \ldots, v_{u,m_u}$, and so we again see that vu is in the ideal generated by the set given in item (3.21). ∎

From the claim we see that $\mathrm{in}(I)$ is finitely generated and hence I is finitely generated. It follows that B is finitely presented. By Corollary 2.1, we have that B has a rational Hilbert series with respect to the vector space spanned by elements of item (3.13). ∎

From this we obtain the following corollary.

Corollary 3.1: *The algebra A has a rational Hilbert series with respect to some vector space.*

Let W denote the k-vector space spanned by the elements of $\mathfrak{G}$. By construction, $\Phi(T\mathfrak{G}) \subseteq A$ and it generates A as a k-algebra and has the property that $1_A \in \Phi(T\mathfrak{G})$. From our theorem, the vector space $\overline{W} \subseteq B$ has the property that $\dim(\overline{W}^n) = f(n)$ for some polynomial f, for all n sufficiently large. Since $\overline{W}q$ is a generating subspace for A and q is regular, we have

$$\dim((\overline{W}q)^n) = \dim(\overline{W}^n)$$

and hence A, too, has rational Hilbert series. ∎

We conjecture the following.

CONJECTURE 3.1: *Let A be a finitely generated right Noetherian PI algebra. Then A is finitely presented.*

In the case that A is a finitely generated Noetherian PI ring, it is known (see [7]) that $\mathrm{GKdim}(A) = \mathrm{GKdim}(A/P)$ for some $P \in \mathrm{Spec}(A)$. It follows that A has integer GK dimension in the case that A is a finitely generated Noetherian PI algebra. We therefore make the following conjecture.

CONJECTURE 3.2: *Let A be a finitely generated Noetherian PI algebra. Then A has a rational Hilbert series with respect to some generating vector space V.*

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